

PERTURBATION OF THE WIGNER EQUATION IN INNER PRODUCT C^* -MODULES

JACEK CHMIELIŃSKI, DIJANA ILIŠEVIĆ, MOHAMMAD SAL MOSLEHIAN,
AND GHADIR SADEGHI

ABSTRACT. Let \mathfrak{A} be a C^* -algebra and \mathfrak{B} be a von Neumann algebra that both act on a Hilbert space \mathcal{H} . Let \mathcal{M} and \mathcal{N} be inner product modules over \mathfrak{A} and \mathfrak{B} , respectively. Under certain assumptions we show that for each mapping $f: \mathcal{M} \rightarrow \mathcal{N}$ satisfying

$$\| |\langle f(x), f(y) \rangle| - |\langle x, y \rangle| \| \leq \varphi(x, y) \quad (x, y \in \mathcal{M}),$$

where φ is a control function, there exists a solution $I: \mathcal{M} \rightarrow \mathcal{N}$ of the Wigner equation

$$|\langle I(x), I(y) \rangle| = |\langle x, y \rangle| \quad (x, y \in \mathcal{M})$$

such that

$$\|f(x) - I(x)\| \leq \sqrt{\varphi(x, x)} \quad (x \in \mathcal{M}).$$

1. INTRODUCTION AND PRELIMINARIES

In this paper, we deal with a perturbation of the Wigner equation, establishing a link between two topics: inner product C^* -modules and stability of functional equations.

1.1. Inner product C^* -modules. A C^* -algebra is a Banach $*$ -algebra $(\mathfrak{A}, \|\cdot\|)$ such that $\|a^*a\| = \|a\|^2$ for every $a \in \mathfrak{A}$. Every C^* -algebra can be regarded as a C^* -subalgebra of $\mathbb{B}(\mathcal{H})$, the algebra of all bounded linear operators on some Hilbert space \mathcal{H} . Recall that $a \in \mathfrak{A}$ is called *positive* (we write $a \geq 0$) if $a = b^*$ for some $b \in \mathfrak{A}$. If $a \in \mathfrak{A}$ is positive, then there is a unique positive $b \in \mathfrak{A}$ such that $a = b^2$; such an element b is called the positive square root of a . For every $a \in \mathfrak{A}$, the positive square root of a^*a is denoted by $|a|$.

A *von Neumann algebra* is a C^* -subalgebra of $\mathbb{B}(\mathcal{H})$ which contains the identity operator $id_{\mathcal{H}}$ and is closed in the weak operator topology (the topology generated by the seminorms $a \mapsto |\langle a\xi, \eta \rangle|$, where $\xi, \eta \in \mathcal{H}$). If \mathfrak{A} is a von Neumann algebra,

2000 *Mathematics Subject Classification.* Primary 46L08; Secondary 39B52, 39B82.

Key words and phrases. Wigner equation, inner product C^* -module, stability.

then the following *polar decomposition* holds: for each $a \in \mathfrak{A}$ there exists a partial isometry $u \in \mathfrak{A}$ (i.e., u^*u is a projection) such that $a = u|a|$ and $u^*a = |a|$ (see e.g. [23, Theorem 4.1.10] or [11, Theorem I.8.1]).

Let $(\mathfrak{A}, \|\cdot\|)$ be a C^* -algebra and let \mathcal{X} be an algebraic right \mathfrak{A} -module which is a complex linear space with $(\lambda x)a = x(\lambda a) = \lambda(xa)$ for all $x \in \mathcal{X}$, $a \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. The space \mathcal{X} is called a *(right) inner product \mathfrak{A} -module* (*inner product C^* -module over the C^* -algebra \mathfrak{A} , pre-Hilbert \mathfrak{A} -module*) if there exists an \mathfrak{A} -valued inner product, i.e., a mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \rightarrow \mathfrak{A}$ satisfying

- (i) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ if and only if $x = 0$,
- (ii) $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$,
- (iii) $\langle x, ya \rangle = \langle x, y \rangle a$,
- (iv) $\langle y, x \rangle = \langle x, y \rangle^*$,

for all $x, y, z \in \mathcal{X}$, $a \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. The conditions (ii) and (iv) yield the fact that the inner product is conjugate-linear with respect to the first variable. Elements $x, y \in \mathcal{X}$ are called orthogonal if and only if $\langle x, y \rangle = 0$. In an inner product \mathfrak{A} -module \mathcal{X} the following version of the *Cauchy-Schwarz inequality* is true:

$$\|\langle x, y \rangle\| \leq \|x\|_{\mathcal{X}} \|y\|_{\mathcal{X}} \quad (x, y \in \mathcal{X}),$$

where $\|x\|_{\mathcal{X}} = \sqrt{\|\langle x, x \rangle\|}$ for all $x \in \mathcal{X}$ (see e.g. [21, Proposition 1.2.4.(iii)]). It follows that $\|\cdot\|_{\mathcal{X}}$ is a norm on \mathcal{X} , so $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a normed space (we will denote the norm in \mathcal{X} simply by $\|\cdot\|$, omitting the subscript \mathcal{X} ; it will be clear from the context whether $\|\cdot\|$ denotes the norm on \mathfrak{A} or the norm on \mathcal{X}). If this normed space is complete, then \mathcal{X} is called a *Hilbert \mathfrak{A} -module*, or a *Hilbert C^* -module over the C^* -algebra \mathfrak{A}* . A left inner product \mathfrak{A} -module can be defined analogously. Any inner product (resp. Hilbert) space is an inner product (resp. Hilbert) \mathbb{C} -module and any C^* -algebra \mathfrak{A} is a Hilbert C^* -module over itself via $\langle a, b \rangle = a^*b$, for all $a, b \in \mathfrak{A}$.

For an inner product \mathfrak{A} -module \mathcal{X} , let $\mathcal{X}^\#$ be the set of all bounded \mathfrak{A} -linear mappings from \mathcal{X} into \mathfrak{A} , that is, the set of all bounded linear mappings $f : \mathcal{X} \rightarrow \mathfrak{A}$ such that $f(xa) = f(x)a$ for all $x \in \mathcal{X}$, $a \in \mathfrak{A}$. Every $x \in \mathcal{X}$ gives rise to a mapping $\hat{x} \in \mathcal{X}^\#$ defined by $\hat{x}(y) = \langle x, y \rangle$ for all $y \in \mathcal{X}$. A Hilbert module \mathcal{X} is called *self-dual* if $\mathcal{X}^\# = \{\hat{x} : x \in \mathcal{X}\}$.

More information on inner product modules can be found e.g. in monographs [20, 21].

1.2. Stability of functional equations. Defining, in some way, the class of approximate solutions of the given functional equation, one can ask whether each mapping from this class can be somehow approximated by an exact solution of the considered equation. Such a problem was formulated by Ulam in 1940 (cf. [27]) and solved in the next year for the Cauchy functional equation by Hyers [14]. It gave rise to the *stability theory* for functional equations. Subsequently, various approaches to the problem have been introduced by several authors. For the history and various aspects of this theory we refer the reader to monographs [15, 17]. Recently, the stability problems have been investigated in Hilbert C^* -modules as well; see [1, 2].

1.3. Wigner equation. We will be considering the *Wigner equation*

$$|\langle I(x), I(y) \rangle| = |\langle x, y \rangle| \quad (x, y \in \mathcal{M}), \quad (\text{W})$$

where $I: \mathcal{M} \rightarrow \mathcal{N}$ is a mapping between inner product modules \mathcal{M} and \mathcal{N} over certain C^* -algebras.

We say that two mappings $f, g: \mathcal{M} \rightarrow \mathcal{N}$ are *phase-equivalent* if and only if there exists a scalar valued mapping $\xi: \mathcal{M} \rightarrow \mathbb{C}$ such that $|\xi(x)| = 1$ and $f(x) = \xi(x)g(x)$ for all $x \in \mathcal{M}$. The equation (W) has been already introduced in 1931 by E.P. Wigner [28] in the realm of (complex) Hilbert spaces. The classical Wigner's theorem, stating that a solution of (W) has to be phase-equivalent to a unitary or antiunitary operator, has deep applications in physics, see [24, 26]. One of the proofs of this remarkable result can be found e.g. in [13] (for further comments we refer also to [25]). Recently, Wigner's result has been studied in the realm of Hilbert modules (cf. e.g. [3, 4, 22]). The stability of the Wigner equation has been extensively studied for Hilbert spaces only (cf. a survey paper [8] or [15, Chapter 9]).

In the following section we consider the stability of the Wigner equation in the setting of inner product modules. Let us mention that the stability of the related *orthogonality equation*

$$\langle I(x), I(y) \rangle = \langle x, y \rangle$$

in this framework has been recently established in [10].

2. STABILITY OF THE WIGNER EQUATION

Suppose that we are given a *control mapping* $\varphi: \mathcal{M} \times \mathcal{M} \rightarrow [0, \infty)$ satisfying, with some constant $0 < c \neq 1$, the following pointwise convergence and boundedness:

- (a) $\lim_{n \rightarrow \infty} c^n \varphi(c^{-n}x, y) = 0$ and $\lim_{n \rightarrow \infty} c^n \varphi(x, c^{-n}y) = 0$ for any fixed $x, y \in \mathcal{M}$;
 (b) the sequence $(c^{2n} \varphi(c^{-n}x, c^{-n}x))$ is bounded for any fixed $x \in \mathcal{M}$.

(2.1)

We say that a mapping $f: \mathcal{M} \rightarrow \mathcal{N}$ *approximately* satisfies the Wigner equation if

$$\| |\langle f(x), f(y) \rangle| - |\langle x, y \rangle| \| \leq \varphi(x, y) \quad (x, y \in \mathcal{M}). \quad (\text{W}_\varphi)$$

The question we would like to answer is if each solution of (W_φ) can be approximated by a solution of (W) .

Let us consider the following condition on an inner product C^* -module \mathcal{X} .

- [H] For each norm-bounded sequence (x_n) in \mathcal{X} , there exists a subsequence (x_{l_n}) of (x_n) and $x_0 \in \mathcal{X}$ such that

$$\| \langle x_{l_n}, y \rangle - \langle x_0, y \rangle \| \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad \text{for all } y \in \mathcal{X}.$$

Validity of [H] in Hilbert spaces follows from its reflexivity and the fact that each ball is sequentially weakly compact. It is an interesting question to characterize the class of all inner product C^* -modules in which [H] is satisfied. In [12] the following similar condition is considered.

- [F] The unit ball of \mathcal{X} is complete with respect to the topology which is induced by the semi-norms $x \mapsto \| \langle x, y \rangle \|$ with $y \in \mathcal{X}$, $\|y\| \leq 1$.

We have the following result.

Proposition 2.1. *If \mathcal{X} is a Hilbert C^* -module over a finite-dimensional C^* -algebra \mathfrak{A} , then the condition [H] is satisfied.*

Proof. Let (x_n) be a bounded sequence in \mathcal{X} ($\|x_n\| \leq M$, $n = 1, 2, \dots$). Then the \mathfrak{A} -valued sequence $(\langle x_n, x_1 \rangle)$ is bounded ($\|\langle x_n, x_1 \rangle\| \leq M \|x_1\|$). Since \mathfrak{A} is finite-dimensional, the theorem of Bolzano-Weierstrass holds true, so there exists a subsequence $(\langle x_n^1, x_1 \rangle)$ of the sequence $(\langle x_n, x_1 \rangle)$ convergent in \mathfrak{A} . Next, we may choose, by the same reason, a convergent subsequence $(\langle x_n^2, x_2 \rangle)$ of the bounded sequence $(\langle x_n^1, x_2 \rangle)$, and so on. Define $x_{l_n} := x_n^n$. Obviously, (x_{l_n})

is a subsequence of (x_n) . It is also clear from the construction of (x_{l_n}) that $\langle x_{l_n}, x_i \rangle = \langle x_n^n, x_i \rangle$ is convergent in \mathfrak{A} for $i = 1, 2, \dots$ (when $n \rightarrow \infty$). Therefore also $\langle x_{l_n}, z \rangle$ is convergent for all z in \mathcal{X}_0 — the closed submodule of \mathcal{X} generated by the sequence $\{x_1, x_2, \dots\}$. Since \mathfrak{A} is finite-dimensional, the Hilbert \mathfrak{A} -module \mathcal{X}_0 is self-dual (cf. [21, p. 27]) and we have $\mathcal{X} = \mathcal{X}_0 \oplus \mathcal{X}_0^\perp$ (cf. [21, Proposition 2.5.4]). For any $y \in \mathcal{X}$, $y = z + z'$ with some $z \in \mathcal{X}_0$ and $z' \in \mathcal{X}_0^\perp$. Thus $\langle x_{l_n}, y \rangle = \langle x_{l_n}, z \rangle + \langle x_{l_n}, z' \rangle = \langle x_{l_n}, z \rangle$, whence $\langle x_{l_n}, y \rangle$ is convergent to some $\varphi(y)$ in \mathfrak{A} for any $y \in \mathcal{X}$. The mapping φ is \mathfrak{A} -linear. Moreover, we have $\|\langle x_{l_n}, y \rangle\| \leq \|x_{l_n}\| \|y\| \leq M \|y\|$, whence $\|\varphi(y)\| \leq M \|y\|$ and thus $\varphi \in \mathcal{X}^\#$. From the self-duality of \mathcal{X} there exists $x_0 \in \mathcal{X}$ such that $\varphi(y) = \langle x_0, y \rangle$ for all $y \in \mathcal{X}$. From the definition of φ this means $\langle x_{l_n}, y \rangle \rightarrow \langle x_0, y \rangle$ for all $y \in \mathcal{X}$. \square

Theorem 2.2. *Let \mathfrak{A} be a C^* -algebra and \mathfrak{B} be a von Neumann algebra that both act on a Hilbert space \mathcal{H} . Let \mathcal{M} be an inner product \mathfrak{A} -module and let \mathcal{N} be an inner product \mathfrak{B} -module satisfying [H]. Then, for each mapping $f: \mathcal{M} \rightarrow \mathcal{N}$ satisfying (W_φ) , with φ satisfying (2.1), there exists $I: \mathcal{M} \rightarrow \mathcal{N}$ with the following properties:*

- (i) $\langle I(x), I(x) \rangle = \langle x, x \rangle \quad (x \in \mathcal{M})$,
- (ii) I preserves orthogonality in both directions, that is, $\langle x, y \rangle = 0$ if and only if $\langle I(x), I(y) \rangle = 0$,
- (iii) $\|f(x) - I(x)\| \leq \sqrt{\varphi(x, x)} \quad (x \in \mathcal{M})$.

Furthermore, there exists $h: \mathcal{M} \rightarrow \mathcal{N}$ such that the following decomposition holds:

$$f(x) = h(x) + I(x), \quad \langle h(x), I(x) \rangle = 0 \quad \text{and} \quad \|h(x)\| \leq \sqrt{\varphi(x, x)} \quad (x \in \mathcal{M}).$$

If \mathfrak{B} is abelian, then I can be chosen as a solution of (W).

Proof. For $n \in \mathbb{N} \cup \{0\}$ let $f_n(x) := c^n f(c^{-n}x)$, $x \in \mathcal{M}$. Substituting in (W_φ) , $c^{-n}x$ and $c^{-m}y$ (with $m, n \in \mathbb{N} \cup \{0\}$) for x and y , respectively, one obtains

$$\| |\langle f_n(x), f_m(y) \rangle| - |\langle x, y \rangle| \| \leq c^{m+n} \varphi(c^{-n}x, c^{-m}y) \quad (x, y \in \mathcal{M}). \quad (2.2)$$

By (2.1-a),

$$\| |\langle f_n(x), f(y) \rangle| - |\langle x, y \rangle| \| \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad (x, y \in \mathcal{M}),$$

which, by the continuity of multiplication, implies

$$\| |\langle f_n(x), f(y) \rangle|^2 - |\langle x, y \rangle|^2 \| \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad (x, y \in \mathcal{M}). \quad (2.3)$$

Analogously,

$$\| |\langle f(x), f_n(y) \rangle| - |\langle x, y \rangle| \| \rightarrow 0 \text{ (as } n \rightarrow \infty) \quad (x, y \in \mathcal{M})$$

implies

$$\| |\langle f(x), f_n(y) \rangle|^2 - |\langle x, y \rangle|^2 \| \rightarrow 0 \text{ (as } n \rightarrow \infty) \quad (x, y \in \mathcal{M}). \quad (2.4)$$

For $x = y$ and $n = m$, (2.2) yields

$$\| \langle f_n(x), f_n(x) \rangle - \langle x, x \rangle \| \leq c^{2n} \varphi(c^{-n}x, c^{-n}x) \quad (x \in \mathcal{M}). \quad (2.5)$$

Then we have

$$\| \langle f_n(x), f_n(x) \rangle \| - \| \langle x, x \rangle \| \leq \| \langle f_n(x), f_n(x) \rangle - \langle x, x \rangle \| \leq c^{2n} \varphi(c^{-n}x, c^{-n}x)$$

for all $x \in \mathcal{M}$, whence

$$\| f_n(x) \|^2 \leq \| x \|^2 + c^{2n} \varphi(c^{-n}x, c^{-n}x) \quad (x \in \mathcal{M}).$$

Let us fix $x \in \mathcal{M}$. By (2.1-b), the sequence $(f_n(x))$ in \mathcal{N} is norm-bounded and therefore, due to [H], there exists a subsequence of $(f_n(x))$ (for simplicity we shall assume that $(f_n(x))$ has such a property) and $F(x) \in \mathcal{N}$ such that

$$\| \langle f_n(x), v \rangle - \langle F(x), v \rangle \| \rightarrow 0 \text{ (as } n \rightarrow \infty) \quad (v \in \mathcal{N}).$$

By the continuity of multiplication and the continuity of involution $*$, this yields

$$\| |\langle f_n(x), v \rangle|^2 - |\langle F(x), v \rangle|^2 \| \rightarrow 0 \text{ (as } n \rightarrow \infty) \quad (v \in \mathcal{N}), \quad (2.6)$$

as well as

$$\| |\langle v, f_n(x) \rangle|^2 - |\langle v, F(x) \rangle|^2 \| \rightarrow 0 \text{ (as } n \rightarrow \infty) \quad (v \in \mathcal{N}). \quad (2.7)$$

Thus we have defined the mapping $F : \mathcal{M} \rightarrow \mathcal{N}$ such that (2.6) and (2.7) are true for each $x \in \mathcal{M}$. In particular, (2.6) implies

$$\| |\langle f_n(x), f(y) \rangle|^2 - |\langle F(x), f(y) \rangle|^2 \| \rightarrow 0 \text{ (as } n \rightarrow \infty) \quad (x, y \in \mathcal{M}).$$

Hence, because of (2.3),

$$|\langle F(x), f(y) \rangle|^2 = |\langle x, y \rangle|^2 \quad (x, y \in \mathcal{M}). \quad (2.8)$$

Inserting $c^{-n}y$ instead of y , we obtain

$$|\langle F(x), f_n(y) \rangle|^2 = |\langle x, y \rangle|^2 \quad (x, y \in \mathcal{M}). \quad (2.9)$$

Letting $n \rightarrow \infty$ this yields

$$|\langle F(x), F(y) \rangle|^2 = |\langle x, y \rangle|^2 \quad (x, y \in \mathcal{M}),$$

and finally

$$|\langle F(x), F(y) \rangle| = |\langle x, y \rangle| \quad (x, y \in \mathcal{M}). \quad (2.10)$$

In particular,

$$\langle F(x), F(x) \rangle = \langle x, x \rangle \quad (x \in \mathcal{M}).$$

Note that (2.8) implies

$$|\langle F(x), f(x) \rangle|^2 = \langle x, x \rangle^2 \quad (x \in \mathcal{M}). \quad (2.11)$$

From (2.7) we get

$$\| |\langle f(x), f_n(x) \rangle|^2 - |\langle f(x), F(x) \rangle|^2 \| \rightarrow 0 \quad (\text{as } n \rightarrow \infty) \quad (x \in \mathcal{M}),$$

which implies, because of (2.4),

$$|\langle f(x), F(x) \rangle|^2 = \langle x, x \rangle^2 \quad (x \in \mathcal{M}). \quad (2.12)$$

Comparing (2.11) and (2.12) we conclude

$$|\langle F(x), f(x) \rangle|^2 = |\langle f(x), F(x) \rangle|^2 \quad (x \in \mathcal{M}).$$

Hence, $\langle F(x), f(x) \rangle$ is a normal element in \mathfrak{B} for every $x \in \mathcal{M}$. Let us fix an arbitrary $x \in \mathcal{M}$. Let $\mathfrak{B}(x)$ be the von Neumann algebra generated by the set $\{\langle F(x), f(x) \rangle, \langle f(x), F(x) \rangle, id_{\mathcal{H}}\}$. Then $\mathfrak{B}(x)$ is abelian (cf. e.g. [23, p. 117]) and $\mathfrak{B}(x) \subseteq \mathfrak{B}$.

Using the polar decomposition we can find a partial isometry $s(x) \in \mathfrak{B}(x)$ such that

$$s(x)|\langle F(x), f(x) \rangle| = \langle F(x), f(x) \rangle \quad \text{and} \quad s(x)^* \langle F(x), f(x) \rangle = |\langle F(x), f(x) \rangle|.$$

Since $|\langle F(x), f(x) \rangle| = \langle x, x \rangle$, this can be written as

$$s(x)\langle x, x \rangle = \langle F(x), f(x) \rangle \quad \text{and} \quad s(x)^* \langle F(x), f(x) \rangle = \langle x, x \rangle.$$

In particular,

$$s(x)^* s(x) \langle x, x \rangle = s(x)^* \langle F(x), f(x) \rangle = \langle x, x \rangle.$$

Since $\mathfrak{B}(x)$ is abelian and $\langle x, x \rangle = |\langle F(x), f(x) \rangle| \in \mathfrak{B}(x)$, we conclude that all elements in $\mathfrak{B}(x)$ commute with $\langle x, x \rangle$. If we define $p(x) = s(x)^* s(x)$, then $p(x)$ is a projection in $\mathfrak{B}(x)$ such that

$$p(x)\langle x, x \rangle = \langle x, x \rangle p(x) = \langle x, x \rangle.$$

Since $\langle F(x), F(x) \rangle = \langle x, x \rangle$, this implies

$$\begin{aligned}
\langle F(x)p(x) - F(x), F(x)p(x) - F(x) \rangle &= p(x)\langle F(x), F(x) \rangle p(x) \\
&\quad - p(x)\langle F(x), F(x) \rangle \\
&\quad - \langle F(x), F(x) \rangle p(x) \\
&\quad + \langle F(x), F(x) \rangle \\
&= p(x)\langle x, x \rangle p(x) - p(x)\langle x, x \rangle \\
&\quad - \langle x, x \rangle p(x) + \langle x, x \rangle = 0.
\end{aligned}$$

Thus

$$F(x)p(x) = F(x) \quad (x \in \mathcal{M}). \quad (2.13)$$

Let us define

$$I(x) = F(x)s(x) \in \mathcal{N}.$$

Then

$$\langle I(x), f(x) \rangle = s(x)^* \langle F(x), f(x) \rangle = \langle x, x \rangle,$$

whence by taking the adjoint,

$$\langle f(x), I(x) \rangle = \langle x, x \rangle. \quad (2.14)$$

We have defined a mapping $I: \mathcal{M} \rightarrow \mathcal{N}$. We will show that it satisfies the desired properties.

First we have, for all $x \in \mathcal{M}$,

$$\begin{aligned}
\langle I(x), I(x) \rangle &= s(x)^* \langle F(x), F(x) \rangle s(x) \\
&= s(x)^* \langle x, x \rangle s(x) = s(x)^* s(x) \langle x, x \rangle \\
&= p(x) \langle x, x \rangle = \langle x, x \rangle.
\end{aligned} \quad (2.15)$$

This implies

$$\begin{aligned}
\langle f(x) - I(x), f(x) - I(x) \rangle &= \langle f(x), f(x) \rangle - \langle f(x), I(x) \rangle \\
&\quad - \langle I(x), f(x) \rangle + \langle I(x), I(x) \rangle \\
&= \langle f(x), f(x) \rangle - \langle x, x \rangle - \langle x, x \rangle + \langle x, x \rangle \\
&= \langle f(x), f(x) \rangle - \langle x, x \rangle,
\end{aligned}$$

which yields

$$\|f(x) - I(x)\|^2 = \|\langle f(x), f(x) \rangle - \langle x, x \rangle\| \leq \varphi(x, x) \quad (x \in \mathcal{M})$$

and finally

$$\|f(x) - I(x)\| \leq \sqrt{\varphi(x, x)} \quad (x \in \mathcal{M}).$$

If $\langle x, y \rangle = 0$, then (2.10) implies $\langle F(x), F(y) \rangle = 0$. This yields $\langle I(x), I(y) \rangle = s(x)^* \langle F(x), F(y) \rangle s(y) = 0$.

Since $s(x) \in \mathfrak{B}(x)$ and $\mathfrak{B}(x)$ is an abelian von Neumann algebra, $s(x)s(x)^* = s(x)^*s(x) = p(x)$, so

$$F(x) = F(x)p(x) = F(x)s(x)s(x)^* = I(x)s(x)^*.$$

If $\langle I(x), I(y) \rangle = 0$, then $\langle F(x), F(y) \rangle = s(x)\langle I(x), I(y) \rangle s(y)^* = 0$ and (2.10) implies $\langle x, y \rangle = 0$. Hence, I preserves orthogonality in both directions.

If we define $h(x) := f(x) - I(x)$ for all $x \in \mathcal{M}$, then, by (2.14) and (2.15),

$$\langle h(x), I(x) \rangle = \langle f(x), I(x) \rangle - \langle I(x), I(x) \rangle = \langle x, x \rangle - \langle x, x \rangle = 0.$$

Now assume that \mathfrak{B} is abelian. According to (2.10) we have

$$\begin{aligned} |\langle I(x), I(y) \rangle|^2 &= s(y)^* \langle F(y), F(x) \rangle s(x)s(x)^* \langle F(x), F(y) \rangle s(y) \\ &= p(y) \langle F(y), F(x) \rangle p(x) \langle F(x), F(y) \rangle \\ &= \langle F(y)p(y), F(x)p(x) \rangle \langle F(x), F(y) \rangle \\ &= |\langle F(x), F(y) \rangle|^2 = |\langle x, y \rangle|^2 \quad (x, y \in \mathcal{M}), \end{aligned}$$

so I is a solution of (W). □

Remark 2.3. The conclusion (i) of Theorem 2.2 implies $\langle x, x \rangle = \langle I(x), I(x) \rangle \in \mathfrak{B}$. Since $\langle x, x \rangle \in \mathfrak{B}$ for all $x \in \mathcal{M}$, we have $\langle x, y \rangle = 1/4 \sum_{i=0}^3 i^k \langle x + i^k y, x + i^k y \rangle \in \mathfrak{B}$ for all $x, y \in \mathcal{M}$. If \mathcal{M} is a full Hilbert \mathfrak{A} -module (that is, the ideal $\langle \mathcal{M}, \mathcal{M} \rangle$ generated by all products $\langle x, y \rangle$, $x, y \in \mathcal{M}$, is dense in \mathfrak{A}), this yields $\mathfrak{A} \subseteq \mathfrak{B}$.

3. APPLICATIONS AND NOTES

The *orthogonality preserving property* of the mapping I in Theorem 2.2 leads to a question on how this property is related to the Wigner equation. Such a relation was shown in the realm of Hilbert spaces and under some linearity-type assumptions (cf. e.g. [7, Corollary 2.4]). Linear orthogonality preserving mappings have also been studied in normed spaces, with the Birkhoff-James or the semi-inner product orthogonalities (cf. [18, 19, 5, 9]). It seems that the investigation on such a class of mappings between inner product modules would be of independent interest; some recent results in that direction can be found in [16].

The following result is a consequence of Theorem 2.2 and Proposition 2.1.

Corollary 3.1. *Let \mathfrak{A} and \mathfrak{B} be finite-dimensional C^* -algebras acting on a Hilbert space \mathcal{H} and let \mathfrak{B} be abelian and containing the identity operator $\text{id}_{\mathcal{H}}$. Let \mathcal{M} be an inner product \mathfrak{A} -module and let \mathcal{N} be a Hilbert \mathfrak{B} -module. Then, for each mapping $f: \mathcal{M} \rightarrow \mathcal{N}$ satisfying (W_{φ}) , with φ satisfying (2.1), there exists a solution $I: \mathcal{M} \rightarrow \mathcal{N}$ of (W) such that*

$$\|f(x) - I(x)\| \leq \sqrt{\varphi(x, x)} \quad (x \in \mathcal{M}).$$

In particular, for $\mathfrak{A} = \mathfrak{B} = \mathbb{C}$ we obtain the stability of the Wigner equation between an inner product space \mathcal{M} and a Hilbert space \mathcal{N} .

One can consider the control mapping

$$\varphi(x, y) := \varepsilon \|x\|^p \|y\|^q,$$

where $\varepsilon > 0$ and p, q are fixed real numbers such that either $p, q > 1$ or $p, q < 1$ (we assume $\|0\|^0 = 1$ and $\|0\|^p = \infty$ for $p < 0$). It is easy to check that the conditions (2.1) are satisfied with $c = 2$ (if $p, q > 1$) or $c = \frac{1}{2}$ (if $p, q < 1$). Then the results from the previous section yield the following result.

Corollary 3.2. *Let \mathfrak{A} be a C^* -algebra and \mathfrak{B} be a von Neumann algebra that both act on a Hilbert space \mathcal{H} . Let \mathcal{M} be an inner product \mathfrak{A} -module and let \mathcal{N} be an inner product \mathfrak{B} -module satisfying [H]. Let either $p, q > 1$ or $p, q < 1$ and $\varepsilon > 0$. Then, for each mapping $f: \mathcal{M} \rightarrow \mathcal{N}$ satisfying*

$$\| |\langle f(x), f(y) \rangle| - |\langle x, y \rangle| \| \leq \varepsilon \|x\|^p \|y\|^q \quad (x, y \in \mathcal{M}),$$

there exists $I: \mathcal{M} \rightarrow \mathcal{N}$ with the following properties:

- (i) $\langle I(x), I(x) \rangle = \langle x, x \rangle \quad (x \in \mathcal{M}),$
- (ii) I preserves orthogonality in both directions,
- (iii) $\|f(x) - I(x)\| \leq \sqrt{\varepsilon} \|x\|^{(p+q)/2} \quad (x \in \mathcal{M}).$

Moreover, if \mathfrak{B} is abelian, then I can be chosen as a solution of (W).

Remark 3.3. The case when the control mapping is of the form $\varphi(x, y) = \varepsilon \|x\| \|y\|$ is still unsolved even in the framework of Hilbert spaces (cf. [6]).

As a consequence of Theorem 2.2, we have obtained the already known stability results in the category of Hilbert spaces. However, in Hilbert spaces it has been proved (cf. [8]) that the mapping I appearing in the assertion of Theorem 2.2 is unique up to phase-equivalence. The proof used, in particular, the Wigner's theorem and the fact that the equality in Cauchy-Schwarz inequality yields linear

dependence of vectors. However, in the setting of Hilbert C^* -modules Wigner's theorem has not been established generally (it has been investigated only for some specific classes of C^* -algebras, e.g. [3, 4]). The uniqueness of I in the assertion of Theorem 2.2 remains an open problem.

Acknowledgments. The authors thank the anonymous referee for his/her remarks; in particular for paying their attention to the paper [12].

The second author was in part supported by the Ministry of Science, Education and Sports of the Republic of Croatia (project No. 037-0372784-2757).

REFERENCES

- [1] M. Amyari, Stability of C^* -inner products, *J. Math. Anal. Appl.* **322** (2006), 214–218.
- [2] M. Amyari, M.S. Moslehian, Stability of derivations on Hilbert C^* -modules, *Contemp. Math.* **427** (2007), 31–39.
- [3] D. Bakić, B. Guljaš, Wigner's theorem in Hilbert C^* -modules over C^* -algebras of compact operators, *Proc. Amer. Math. Soc.* **130** (2002), 2343–2349.
- [4] D. Bakić, B. Guljaš, Wigner's theorem in a class of Hilbert C^* -modules, *J. Math. Phys.* **44** (2003), 2186–2191.
- [5] A. Blanco, A. Turnšek, On maps that preserve orthogonality in normed spaces, *Proc. Roy. Soc. Edinburgh Sect. A* **136** (2006), 709–716.
- [6] J. Chmieliński, On a singular case in the Hyers–Ulam–Rassias stability of the Wigner equation, *J. Math. Anal. Appl.* **289** (2004), 571–583.
- [7] J. Chmieliński, Orthogonality preserving property, Wigner equation and stability, *J. Inequal. Appl.* vol. **2006** (2006), Article ID 76489, 1–9.
- [8] J. Chmieliński, Stability of the Wigner equation and related topics, *Nonlinear Funct. Anal. Appl.* **11** (2006), 859–879.
- [9] J. Chmieliński, Remarks on orthogonality preserving mappings in normed spaces and some stability problems, *Banach J. Math. Anal.* **1** (2007), 117–124.
- [10] J. Chmieliński, M.S. Moslehian, Approximately C^* -inner product preserving mappings, *Bull. Korean Math. Soc.* (to appear).
- [11] K.R. Davidson, C^* -algebras by Example, Fields Institute Monographs, 6. American Mathematical Society, Providence, RI, 1996.
- [12] M. Frank, Geometrical aspects of Hilbert C^* -modules, *Positivity* **3** (1999), 215–243.
- [13] M. Györy, A new proof of Wigner's theorem, *Rep. Math. Phys.* **54** (2004), 159–167.
- [14] D.H. Hyers, On the stability of the linear functional equation, *Proc. Natl. Acad. Sci. U.S.A.* **27** (1941), 222–224.
- [15] D.H. Hyers, G. Isac, Th.M. Rassias, Stability of Functional Equations in Several Variables, Progress in Nonlinear Differential Equations and Their Applications vol. 34, Birkhäuser, Boston-Basel-Berlin, 1998.

- [16] D. Ilišević, A. Turnšek, Approximately orthogonality preserving mappings on C^* -modules, *J. Math. Anal. Appl.* **341** (2008), 298–308.
- [17] S.-M. Jung, Hyers–Ulam–Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Inc., Palm Harbor, 2001.
- [18] D. Koehler, P. Rosenthal, On isometries of normed linear spaces, *Studia Math.* **36** (1970), 213–216.
- [19] A. Koldobsky, Operators preserving orthogonality are isometries, *Proc. Roy. Soc. Edinburgh Sect. A* **123** (1993), 835–837.
- [20] E.C. Lance, Hilbert C^* -Modules, London Mathematical Society Lecture Note Series, 210, Cambridge University Press, 1995.
- [21] V.M. Manuilov, E.V. Troitsky, Hilbert C^* -Modules, Translations of Mathematical Monographs, 226. American Mathematical Society, Providence, RI, 2005.
- [22] L. Molnár, A generalization of Wigner’s unitary-antiunitary theorem to Hilbert modules, *J. Math. Phys.* **40** (1999), 5544–5554.
- [23] J.G. Murphy, C^* -Algebras and Operator Theory, Academic Press, Boston, 1990.
- [24] N.E. Poschadel, A note on families of spin observables and a generalization of Wigner’s theorem in \mathbf{C}^2 , *J. Math. Phys.* **41** (2000), no. 11, 7832–7838.
- [25] J. Rätz, On Wigner’s theorem: Remarks, complements, comments, and corollaries, *Aequationes Math.* **52** (1996), 1–9.
- [26] S. Schreckenberg, Symmetry and history quantum theory: an analog of Wigner’s theorem, *J. Math. Phys.* **37** (1996), no. 12, 6086–6105.
- [27] S.M. Ulam, Problems in Modern Mathematics (Chapter VI, Some Questions in Analysis: §1, Stability), Science Editions, John Wiley & Sons, New York, 1964.
- [28] E.P. Wigner, Gruppentheorie und ihre Anwendungen auf die Quantenmechanik der Atom-spektren, Friedr. Vieweg und Sohn, Braunschweig, 1931.

INSTITUTE OF MATHEMATICS, PEDAGOGICAL UNIVERSITY OF CRACOW, PODCHORĄŻYCH
2, 30-084 KRAKÓW, POLAND

E-mail address: jacek@ap.krakow.pl

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ZAGREB, BIJENIČKA 30, P.O. Box 335,
10 002 ZAGREB, CROATIA

E-mail address: ilisevic@math.hr

DEPARTMENT OF MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. Box 1159,
MASHHAD 91775, IRAN;

CENTRE OF EXCELLENCE IN ANALYSIS ON ALGEBRAIC STRUCTURES (CEAAS), FERDOWSI
UNIVERSITY OF MASHHAD, IRAN.,

E-mail address: moslehian@ferdowsi.um.ac.ir

DEPARTMENT OF MATHEMATICS, FERDOWSI UNIVERSITY OF MASHHAD, P. O. Box 1159,
MASHHAD 91775, IRAN, BANACH MATHEMATICAL RESEARCH GROUP (BMRG), MASHHAD,
IRAN.

E-mail address: ghadir54@yahoo.com